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Let $f:(\mathbf{R}^n,0)\longrightarrow(\mathbf{R},0)$ be a function-germ. We are interested in a relation between the topology of the map f and the mapping degree of a finite map germ determined by f through some procedure.

Let V_ε be a local level manifold of f. i.e. $V_\varepsilon=\tilde{f}^{-1}(\varepsilon)$, where $\tilde{f}:D\longrightarrow\mathbf{R}$ is a representative of f and where D is a small ball in $\mathbf{R}^n$, and its center is the origin of $\mathbf{R}^n$. Abusing the notation, we also denote f by a representative of germ f on a small neighbourhood of the origin. Let $(x_1,...,x_n)$ be a local coordinate system of $\mathbf{R}^n$ at the origin. If f defines an isolated singularity at the origin, then the map

$$df=\left(\frac{\partial f}{\partial x_1},...,\frac{\partial f}{\partial x_n}\right):(\mathbf{R}^n,0)\longrightarrow(\mathbf{R}^n,0)$$

is a finite map. Khimshiashvili(1977), Arnol'd(1977) and Wall(1983) pointed out the following fact:

THEOREM A. $\deg(df)=\operatorname{sgn}(-\varepsilon)^n(1-\chi(V_\varepsilon))$.

COROLLARY. *Combining it with the Eisenbud-Levine's theorem, we obtain an algebraic formula of the Euler number of a local level manifold* V_ε.

COROLLARY. *Let S be the boundary of D, and set $A_+=S\cap\{f\geq 0\}$, $A_-=S\cap\{f\leq 0\}$. Since A_+(resp.A_-) is diffeomorphic to V_ε with $\varepsilon>0$(resp.$\varepsilon<0$), we have*

$$\chi(A_+)=1+(-1)^{n+1}\deg(df),\quad \chi(A_-)=1-\deg(df).$$

Here we will repeat the proof of theorem A, because of its importance. Let δ be a suitably small positive number. By Morse theory, the relative Euler characteristic

$$\chi\left(f^{-1}[-\delta,\delta]\cap D,f^{-1}(-\delta)\cap D\right)=n_+-n_-,$$

where n_+,n_- denote the numbers of critical points of g in $f^{-1}[-\delta,\delta]\cap D$ of even, odd index, or equivalently the number at which dg has local degree $+1, -1$. Here we choose g to be a C^1-approximation to f, whose critical points are non-degenerate. Thus

$$\deg(df) = \deg(dg) = n_+ - n_-.$$

As $f^{-1}[-\delta, \delta] \cap D$ is contractible, so has Euler characteristic 1, we deduce

$$\deg(df) = 1 - \chi\left(f^{-1}(-\delta) \cap D\right) = 1 - \chi(V_{-\delta}).$$

By changing the sign of f, we have

$$(-1)^n \deg(df) = 1 - \chi\left(f^{-1}(\delta) \cap D\right) = 1 - \chi(V_\delta).$$

Next we consider a curve-germ. Say $f : (\mathbf{R}^2, 0) \longrightarrow (\mathbf{R}, 0)$ defines a plane curve-germ. Let (x, y) be a local coordinate system of $\mathbf{R}^2$ at 0, and $g = x^2 + y^2$. The number of connected components of the intersection $f^{-1}(0) \cap \{g > 0\}$ determines topology of $f^{-1}(0)$. Fukuda-Aoki-Sun (1986) proved the following fact.

THEOREM B. *The number of connected components of the intersection* $f^{-1}(0) \cap \{g > 0\}$ *is equal to* $2\deg(j, f)$, *where* $j = \dfrac{\partial(g, f)}{\partial(x, y)}$.

Szafraniec(1988) gave a generalization of this theorem in the following form.

THEOREM C. *Assume that* $f : (\mathbf{R}^n, 0) \longrightarrow (\mathbf{R}^{n-1}, 0)$ *defines a curve-germ, and that* $g : (\mathbf{R}^n, 0) \longrightarrow (\mathbf{R}, 0)$ *has no zero on* $f^{-1}(0) - \{0\}$. *Set*

$b_+ =$ *the number of connected components of* $f^{-1}(0) \cap \{g > 0\}$, *and*
$b_- =$ *the number of connected components of* $f^{-1}(0) \cap \{g < 0\}$.

Then $b_+ - b_- = 2\deg(j, f)$,

where $j = \det\left(\dfrac{\partial g}{\partial x}, \dfrac{\partial f}{\partial x}\right)$, *and* $x = (x_1, ..., x_n)$ *is a local coordinate system of* $\mathbf{R}^n$ *at* 0.

In the case $g = x_1^2 + ... + x_n^2$, this theorem was obtained by Aoki-Fukuda-Nishimura (1987) as a formula of the topological type of $f^{-1}(0)$ in $(\mathbf{R}^n, 0)$. The cases for $g = x_1^2$ and $g = x_1$ were obtained by Nishimura-Aoki-Fukuda (1989) as a formula which determines the bifurcation of 1-parameter family with parameter x_1, of curve-germs $f^{-1}(x_1, 0)$ in $(\mathbf{R}^{n-1}, 0)$. The main idea of this talk is the following: Since

$$b_+ - b_- = 2\{\chi(f^{-1}(\varepsilon) \cap \{g \geq 0\}) - \chi(f^{-1}(\varepsilon) \cap \{g \leq 0\})\},$$

we understand that the above formula asserts that the mapping degree is equal to the difference of Euler numbers. From this point of view, we have a hope to formulate similar theorems for some other map-germs.

We return to function-germ $f : (\mathbf{R}^{n+1}, 0) \longrightarrow (\mathbf{R}, 0)$, where $(\lambda, x_1, ..., x_n)$ is a coordinate system of $\mathbf{R}^{n+1}$ at the origin.

THEOREM D. *If* $\dim_{\mathbf{R}} \mathbf{R}\{\lambda, x\} \Big/ \left(f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right) < \infty$, *then*

$$-\operatorname{sgn}(-\varepsilon)^{n+1} \left\{\chi(f^{-1}(\varepsilon) \cap \{\lambda \geq 0\}) - \chi(f^{-1}(\varepsilon) \cap \{\lambda \leq 0\})\right\} = \deg\left(f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)$$

For a function germ $f : (\mathbf{R}^{n+2}, 0) \longrightarrow (\mathbf{R}, 0)$ with coordinate $(x, y, z_1, \ldots, z_n)$ of $\mathbf{R}^{n+2}$ at 0, we set that $j = \dfrac{\partial(g, f)}{\partial(x, y)}$ for $g = g(x, y)$. Assume that the singular locus of g is in the zero locus of g.

THEOREM E. *If* $\dim_{\mathbf{R}} \mathbf{R}\{x, y, z\} \Big/ \left(f, j, \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\right) < \infty$, *then*

$$-\operatorname{sgn}(-\varepsilon)^{n} \left\{\chi(f^{-1}(\varepsilon) \cap \{g \geq 0\}) - \chi(f^{-1}(\varepsilon) \cap \{g \leq 0\})\right\} = \deg\left(f, j, \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\right)$$

We can understand them as formulae for bifurcation of zeros of functions.

SKETCH OF THE PROOF OF THEOREM E: Replacing f by a suitable perturbation of f, if necessary, we can assume that the restriction of g to $f^{-1}(\varepsilon)$ is a Morse function except the zero locus of g. By Morse theory, the relative Euler characteristic

$$\begin{aligned}
\chi\left(f^{-1}(\varepsilon) \cap \{g \geq 0\}, f^{-1}(\varepsilon) \cap \{g = 0\}\right) &= n_+(g_+) - n_-(g_+) \\
(\text{resp.}\ \chi\left(f^{-1}(\varepsilon) \cap \{g \leq 0\}, f^{-1}(\varepsilon) \cap \{g = 0\}\right) & \\
&= (-1)^{n+1}\left(n_+(g_-) - n_-(g_-)\right)),
\end{aligned}$$

where $n_+(g_+), n_-(g_+)$ (resp. $n_+(g_-), n_-(g_-)$) denote the numbers of critical points of $g|_{f^{-1}(\varepsilon)}$ in $\{g > 0\}$ (resp. $\{g < 0\}$) of even, odd index, or equivalently the number at which $\left(f, j, \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\right)$ has local degree $-\operatorname{sgn}(-\varepsilon)^n$, $\operatorname{sgn}(-\varepsilon)^n$ (resp. $-\operatorname{sgn}\varepsilon^n$, $\operatorname{sgn}\varepsilon^n$). Thus

$$\begin{aligned}
&\deg\left(f, j, \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\right) \\
&= -\operatorname{sgn}(-\varepsilon)^n (n_+(g_+) - n_-(g_+)) - \operatorname{sgn}\varepsilon^n (n_+(g_-) - n_-(g_-)). \\
&= -\operatorname{sgn}(-\varepsilon)^n \left\{\chi(f^{-1}(\varepsilon) \cap \{g \geq 0\}) - \chi(f^{-1}(\varepsilon) \cap \{g \leq 0\})\right\}. \ \blacksquare
\end{aligned}$$

PROOF OF THEOREM E $\Rightarrow$ THEOREM B: Set $n = 0$. ▮

PROOF OF THEOREM E $\Rightarrow$ THEOREM D: Set $j = x$. Then $j = \dfrac{\partial f}{\partial y}$. ▮

PROBLEM: Construct an unified theory that describes the above phenomena.

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